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THE NON-LINEAR OSCILLATIONS OF A SATELLITE WITH THIRD-ORDER RESONANCE†

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Plane non-linear oscillations of an artificial satellite—a rigid body—about its centre of mass in an elliptical orbit of small eccentricity are considered. It is assumed that three times the frequency of small oscillations of the satellite in a circular orbit is close to the frequency of revolution of its centre of mass. Methods of classical perturbation theory are used to reduce the problem to that of a model system, described by a Hamiltonian which is characteristic for problems involving the motion of Hamiltonian systems with one degree of freedom in the case of third-order resonance. A detailed analysis of such systems is carried out. The theory of periodic Poincaré motions and KAM-theory are used to transfer the results for the model system to the complete system and to apply them to the problem of satellite motion. The question of the existence, number and stability of periodic motions with period equal to three times the period of revolution of the centre of mass of the satellite in orbit is considered, depending on the inertial parameter of the satellite and the eccentricity of the orbit. It is shown that motions of the satellite beginning in a certain neighbourhood of its eccentricity oscillations are bounded, and an estimate is given for the size of that neighbourhood.

1. STATEMENT OF THE PROBLEM. TRANSFORMATION OF THE HAMILTONIAN

Plane non-linear oscillations of a satellite, treated as a rigid body, about its centre of mass in an elliptical orbit are described by the equation [1]

$$(1+e\cos\nu)d^2\psi/d\nu^2 - 2e\sin\nu d\psi/d\nu + \omega_0^2\sin\psi\cos\psi = 2e\sin\nu \tag{1.1}$$

where e is the eccentricity of the orbit of the centre of mass, ν is the true anomaly, $\omega_0^2 = 3(C-A)/B$, where A, B and C are the principal central moments of inertia of the body, B being the moment of inertia about an axis perpendicular to the orbital plane and ψ is the angle between the radius vector of the centre of mass of the satellite about the attracting centre and the axis corresponding to the moment of inertia A.

If the eccentricity of the orbit is small and $\omega_0 \neq 1$, Eq. (1.1) has a 2π-periodic solution [1] of the form

$$\psi = \psi^* = \frac{2e\sin\nu}{\omega_0^2-1} + O(e^2) \tag{1.2}$$

which reduces, when $e = 0$, to the solution $\psi^* = 0$ corresponding to equilibrium of the satellite in an orbital system of coordinates. The question of the stability of the eccentricity oscillations (1.2) has been investigated in detail [1, 2].

The aim of this paper is to investigate non-linear oscillations of the satellite in the case when three times the frequency ω_0 of its small oscillations in a circular orbit ($e = 0$) almost equals the frequency of revolution of the centre of mass.

If we put

$$\psi = \psi^* + q/(1+e\cos\nu),\quad p = dq/d\nu$$

the equations of perturbed motion in the neighbourhood of the solution (1.2) may be expressed in Hamiltonian form with Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{e\cos\nu}{2(1+e\cos\nu)}q^2 + \frac{1}{2}\omega_0^2[(1+e\cos\nu)\times$$

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$$\times \sin\left(2\psi^* + \frac{q}{1+e\cos\nu}\right)\sin\frac{q}{1+e\cos\nu} - q\sin 2\psi^*] \tag{1.3}$$

Applying several canonical changes of variables, we reduce the Hamiltonian (1.3) to the form characteristic for problems involving non-linear oscillations of time-periodic Hamiltonian systems with one degree of freedom in the case of third-order resonance considered here [3].

Taking (1.2) into consideration, we can expand the Hamiltonian (1.3) in series

$$H = H_2 + H_3 + H_4 + \dots \tag{1.4}$$

$$H_2 = \frac{1}{2}(p^2 + \omega_0^2 q^2) - \frac{e\cos\nu}{2(\omega_0^2 - 1)}q^2 + O(e^2)$$

$$H_3 = -\frac{2}{3}\frac{e\omega_0^2 \sin\nu}{\omega_0^2 - 1}q^3 + O(e^2), \quad H_4 = H_4^{(0)} + O(e),. \; H_4^{(0)} = -\frac{1}{6}\omega_0^2 q^4$$

where the dots stand for all the terms of order greater than four in q and p.

Applying a linear canonical transformation $q, p \to q_*, p_*$, 2π-periodic in ν, of the form

$$q = \frac{q_*}{\sqrt{\omega_0}} + \frac{e(\cos\nu q_* - 2\omega_0 \sin\nu p_*)}{\sqrt{\omega_0}(\omega_0^2 - 1)(4\omega_0^2 - 1)} + O(e^2)$$

$$p = \sqrt{\omega_0}p_* + \frac{e\sqrt{\omega_0}}{(\omega_0^2 - 1)(4\omega_0^2 - 1)}\left(\frac{2\omega_0^2 - 1}{\omega_0}\sin\nu q_* - \cos\nu p_*\right) + O(e^2)$$

we reduce H_2 to normal form H_{2*}

$$H_{2*} = \tfrac{1}{2}\lambda(q_*^2 + p_*^2), \quad \lambda = \omega_0 + O(e^2)$$

We shall assume that the quantity λ is close to 1/3. In the e, ω_0 plane the relationship $3\lambda = 1$ defines a resonance curve, represented by the solid curve in Fig. 1, whose equation is as follows [2]:

$$\omega_0 = 1/3 + (71/160)e^2 + O(e^4) \tag{1.5}$$

Henceforth we shall put $\lambda = 1/3 + e^2\beta$.

Applying a canonical transformation $q_*, p_* \to x_*, y_*$ of the Birkhoff type, we reduce $H_4^{(0)}$ to normal form $-(x_*^2 + y_*^2)^2/16$. A change of variables $x^3 = e^{-1}x_*, y = e^{-1}y_*$ will then yield the following Hamiltonian

$$K = \frac{1}{2}\lambda(x^2 + y^2) + \frac{\sqrt{3}}{4}e^2 \sin\nu x^2 - \frac{1}{16}e^2(x^2 + y^2)^2 + O(e^3) \tag{1.6}$$

We transform (1.6) to φ, r coordinates by the formulae $x = \sqrt{(2r)}\sin\varphi$, $y = \sqrt{(2r)}\cos\varphi$, and eliminate summands with non-resonance harmonics in third-order terms by applying a near-identical canonical

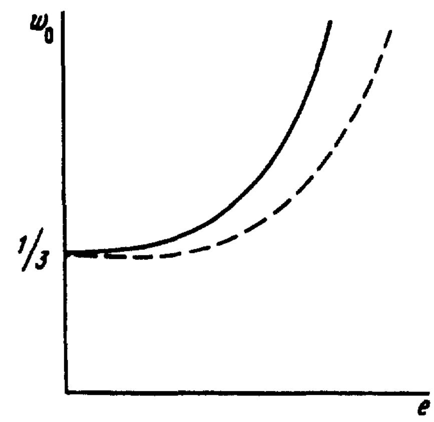

Fig. 1.

transformation, 2π-periodic in ν. This gives the following Hamiltonian

$$\Gamma = \lambda r - \frac{1}{4}e^2r^2 - \frac{\sqrt{6}}{16}e^2r^{3/2}\cos(3\varphi - \nu) + O(e^3) \tag{1.7}$$

Finally, making the change of variable $\varphi, r \to \theta, \rho$, where

$$r = \frac{3}{8}\rho, \quad \varphi = -\theta - \frac{2}{3}\pi + \frac{\nu}{3}$$

and introducing a new independent variable $\tau = (3/32)e^2\nu$, we transform the Hamiltonian (1.7) to the following final form

$$\gamma = \gamma_0 + e\gamma_1\ (\theta, \rho, \tau, e) \tag{1.8}$$

where

$$\gamma_0 = \rho^2 + \rho^{3/2}\cos 3\theta - \mu\rho, \quad \mu = (32/3)\beta \tag{1.9}$$

The function γ_1 in (1.8) is 2π-periodic in θ, periodic in τ with period $T = (9/16)\pi e^2$ and analytic with respect to all the variables in the domain $0 < \rho \ll 1$.

2. INTEGRATION OF THE UNPERTURBED SYSTEM

Let us consider the unperturbed system described by the Hamiltonian (1.9), which is typical for problems involving the motion of Hamiltonian systems with one degree of freedom and third-order resonance. The qualitative nature of non-linear oscillations in such systems, as well as the existence and stability of 6π-periodic motions, was studied in [3–6]. This part of the paper consists of a detailed analytical investigation of a system with Hamiltonian γ_0.

2.1. The equations of motion corresponding to γ_0 are

$$d\theta/d\tau = 2\rho + {}^3\!/_2\sqrt{\rho}\cos 3\theta - \mu, \quad d\rho/d\tau = 3\rho^{3/2}\sin 3\theta \tag{2.1}$$

We will indicate the equilibrium positions of system (2.1) and the nature of their stability [3]. The equilibrium position $\rho = 0$, which exists for any values of μ, is stable when $\mu \neq 0$ and unstable when $\mu = 0$. If $\mu \geqslant -9/32$, there are two further types of equilibrium position, at which $\sin 3\theta = 0$, and the equilibrium values of ρ are the roots $(16\mu + 9\pm 3\sqrt{(9 + 32\mu)})/32$ of a quadratic equation.

The equilibrium positions of system (2.1) corresponding to the larger root (ρ_*) are stable, while those corresponding to the smaller root (ρ_{**}) are unstable. When $\mu = 0$ the unstable equilibrium positions coincide with the origin ($\rho_{**} = 0$). When $\mu = -9/32$ the equation has a double root ($\rho_* = \rho_{**} = 9/64$); the corresponding equilibrium positions are unstable.

Figure 2 shows the qualitatively distinct phase portraits of system (2.1) [3] in the plane of the variables $x_1 = \sqrt{(2r)}\cos\theta$, $x_2 = \sqrt{(2r)}\sin\theta$, for the cases $\mu < -9/32$ (a), $\mu = -9/32$ (b), $-9/32 < \mu < 0$ (c) and $\mu = 0$ (d), $\mu > 0$ (d).

The stable equilibrium positions of system (2.1) are represented in Fig. 2 by singular points of the "centre" type, and unstable ones (for $\mu \neq -9/32$, $\mu \neq 0$) by singular points of the saddle type. When $\mu = 0$ (Fig. 2d) the origin is a compound singular point, whose neighbourhood is the union of six saddle sectors. The unstable equilibrium positions for $\mu = -9/32$ (Fig. 2b) are represented by compound singular points—degenerate saddle-nodes.

The unstable singular points of the system are connected by separatrices, which separate the oscillation domains of the system (near stable equilibria) from domains of rotation. Trajectories in a domain of rotation encircle all the singular points of the system; the angle θ increases monotonically along such trajectories.

2.2. The system of equations (2.1) has a first integral

$$\gamma_0(\theta, \rho) = h = \text{const} \tag{2.2}$$

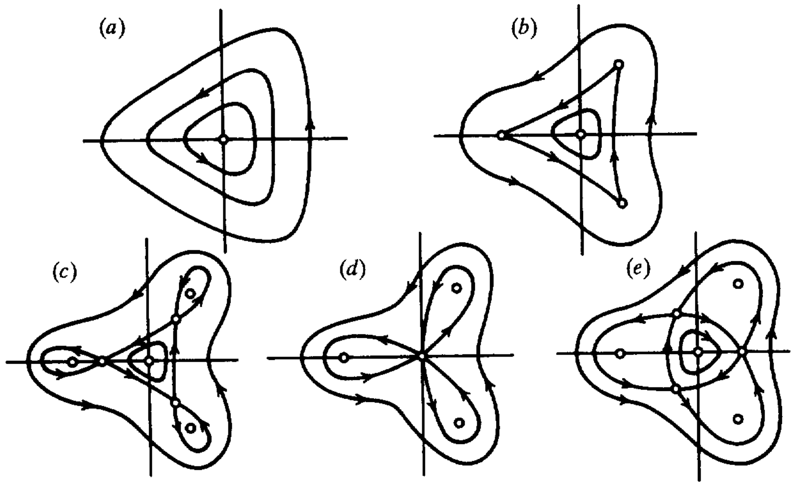

Fig. 2.

using which we can eliminate θ from the second equation of (2.1). Then the equation for ρ becomes

$$d\rho/\sqrt{(F(\rho))} = \pm 3d\tau, \quad F(\rho) = \rho^3 - (h + \mu\rho - \rho^2)^2 \tag{2.3}$$

The upper sign in (2.3) corresponds to motion in the sectors $2\pi k/3 \leqslant \theta \leqslant \pi(2k + 1)/3$ ($k = 0, 1, 2$), where ρ is increasing, and the lower sign, to motion in the sectors $\pi(2k - 1)/3 \leqslant \theta \leqslant 2\pi k/3$ ($k = 1, 2, 3$), where ρ is decreasing.

The form of the solution of Eq. (2.3) depends on the number and multiplicity of real roots of the polynomial $F(\rho)$. It follows from the form of $F(\rho)$ that all its real roots are positive.

After Eq. (2.3) has been integrated, the function $\theta(\tau)$ may be determined from (2.2).

Let us write down the solutions of Eq. (2.3) in the entire range of variation of the parameters μ and h. The eight subdomains indicated in the (μ, h) plane in Fig. 3 correspond to the qualitatively distinct solutions of system (2.1). The subdomains are bounded by the straight lines $\mu = -9/32$, $\mu = 0$, the straight line $h = 0$ and the curves $h = h_1(\mu)$, $h = h_2(\mu)$, corresponding to the energy levels at the equilibrium positions of system (2.1) for which $\rho = 0$ and $\rho = \rho_*$, $\rho = \rho_{**}$, respectively, where $h_1(\mu) = \rho_*^2 + \rho_*^{3/2} - \mu\rho_*$ ($\mu \geqslant -9/32$), $h_2(\mu) = \rho_{**}^2 - \rho_{**}^{3/2} - \mu\rho_{**}$ for $-9/32 \leqslant \mu < 0$ and $h_2(\mu) = \rho_{**}^2 + \rho_{**}^{3/2} - \mu\rho_{**}$ for $\mu \geqslant 0$. The curves $h = h_1(\mu)$ and $h = h_2(\mu)$ issue from a common point with coordinates ($-9/32$, $27/4096$); the curve $h = h_1(\mu)$ and $h = h_2(\mu)$ issue from a common point with coordinates ($-9/32$, $27/4096$); the curve $h = h_1(\mu)$ cuts the Oh axis at $h = -27/256$.

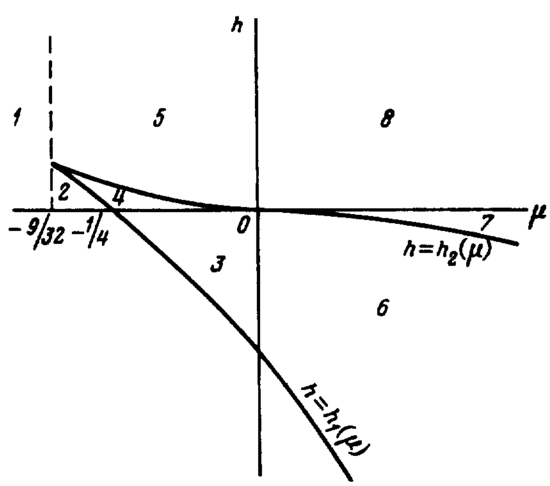

Fig. 3.

No motion is possible if $h < 0$ in the domain $\mu < -1/4$, or if $h < h_1(\mu)$ in the domain $\mu \geqslant -1/4$.

The points of the straight line $h = 0$ ($\mu < -1/4$) represent the stable equilibrium $\rho = 0$ of system (2.1). In domains 1 and 2 (Fig. 3) and the part of the straight line $\mu = -9.32$ separating them we have oscillations near that equilibrium (see Fig. 2a–c), taking place in the interval $\rho_2 \leqslant \rho \leqslant \rho_1$ of the ρ axis, where ρ_1 and ρ_2 ($\rho_2 < \rho_1$) are the real roots of the polynomial $F(\rho)$; the other two roots are complex conjugates. We introduce the notation

$$\alpha = \tfrac{1}{2}(1+2\mu-\rho_1-\rho_2), \quad \beta = h^2/(\rho_1\rho_2)$$
$$\gamma = \beta-\alpha^2, \quad p = [(\alpha-\rho_1)^2+\gamma]^{1/2}, \quad q = [(\alpha-\rho_2)^2+\gamma]^{1/2} \tag{2.4}$$

Using tables of integrals [7], we obtain the following expression from (2.3)

$$\rho(\tau) = \frac{\rho_1 q+\rho_2 p-(\rho_1 q-\rho_2 p)\operatorname{cn}(3\sqrt{pq}\tau)}{p+q+(p-q)\operatorname{cn}(2\sqrt{pq}\tau)} \tag{2.5}$$

The modulus of the elliptic cosine in (2.5) is

$$k_1 = \tfrac{1}{2}\{[(\rho_1-\rho_2)^2-(p-q)^2]/(pq)\}^{1/2} \tag{2.6}$$

and the constant of integration is chosen so that at $\tau = 0$ the variable ρ should achieve its minimum value $\rho = \rho_2$. The frequency of these oscillations is $\omega_1/3$, where

$$\omega_1 = \frac{3\pi\sqrt{pq}}{2K(k_1)} \tag{2.7}$$

and $K(k_1)$ is the complete elliptic integral of the first kind.

The points of the curve $h = h_1(\mu)$ (for $\mu \geqslant -1/4$) represent stable equilibria of system (2.1) corresponding to $\rho = \rho_*$. In domains 3 and 6 and on the part of the straight line $\mu = 0$ separating them (Fig. 3) we have oscillations near these equilibria (see Fig. 2c–e), in which case $\rho_2 \leqslant \rho \leqslant \rho_1$ (ρ_1 and ρ_2 are the real roots of the polynomial $F(\rho)$; its two other roots are complex conjugates). The function $\rho(\tau)$ for these oscillations is defined by Eqs (2.4)–(2.6), and their frequency equals ω_1 (formula (2.7)).

In domain 4 (Fig. 3), there are oscillations of two types corresponding to each value of μ and h: an oscillation of the first type takes place in the neighbourhood of the stable equilibrium $\rho = 0$, and ρ then varies in the range $\rho_4 \leqslant \rho \leqslant \rho_3$; oscillations of the second type take place in the neighbourhood of the stable equilibria corresponding to $\rho = \rho_*$, in which case ρ takes values in the range $\rho_2 \leqslant \rho \leqslant \rho_1$ (see Fig. 2c). Here $\rho_1, \rho_2, \rho_3, \rho_4$, ($\rho_4 < \rho_3 < \rho_2 < \rho_1$) are the real roots of the polynomial $F(\rho)$.

For oscillations of the first type, it follows from Eq. (2.3) that [7]

$$\rho(\tau) = \frac{\rho_4(\rho_1-\rho_3)+\rho_1(\rho_3-\rho_4)\operatorname{sn}^2 u}{\rho_1-\rho_3+(\rho_3-\rho_4)\operatorname{sn}^2 u} \tag{2.8}$$

where we have assumed that $\rho = \rho_4$ at $\tau = 0$, and for oscillations of the second type

$$\rho(\tau) = \frac{\rho_2(\rho_1-\rho_3)-\rho_3(\rho_1-\rho_2)\operatorname{sn}^2 u}{\rho_1-\rho_3-(\rho_1-\rho_2)\operatorname{sn}^2 u} \tag{2.9}$$

where we have assumed that $\rho = \rho_2$ at $\tau = 0$. In both of the last expressions we have introduced the new notation $u = 3/2[(\rho_1-\rho_3)(\rho_2-\rho_4)]^{1/2}$; the modulus of the elliptic function is $k_2 = [(\rho_1-\rho_2)(\rho_3-\rho_4)/(\rho_1-\rho_3)(\rho_2-\rho_4)]^{1/2}$.

Oscillations in the neighbourhood of equilibria corresponding to $\rho = \rho_*$ take place at a frequency

$$\omega_2 = \tfrac{3}{4}\pi K^{-1}(k_2)[(\rho_1-\rho_3)(\rho_2-\rho_4)]^{1/2} \tag{2.10}$$

and those in the neighbourhood of $\rho = 0$ at a frequency $\omega_2/3$.

To each point of domain 7 there correspond two motions (see Fig. 2e): oscillation in the neighbourhood of the stable equilibrium $\rho = 0$, in which case $\rho_4 \leqslant \rho \leqslant \rho_3$, and one of the revolutions for which $\rho_2 \leqslant \rho \leqslant \rho_1$ ($\rho_1, \rho_2, \rho_3, \rho_4$ are the real roots of the polynomial $F(\rho)$, where $\rho_4 < \rho_3 < \rho_2 < \rho_1$). The behaviour of $\rho(\tau)$ for these motions and revolution is given by (2.8) and (2.9), respectively. The oscillation frequency and mean frequency of revolution are equal to $\omega_2/3$, where ω_2 is given by (2.10).

In domains 5 and 8 and on their boundaries—the straight lines $\mu = 0$ and $\mu = -9/32$ (Fig. 3)—we have a revolution for each value of μ and h (see Fig. 2c–e). In these cases $\rho_2 \leqslant \rho \leqslant \rho_1$, where ρ_1 and ρ_2 are the real roots of the polynomial $F(\rho)$ (the other two roots are complex conjugates). The function $\rho(\tau)$ is defined by (2.4)–(2.6), and the mean frequency of revolution is $\omega_1/3$, where ω_1 is given by (2.7).

At the points of the straight line $h = 0$ with $-1/4 < \mu < 0$ (Fig. 3) we have a stable equilibrium $\rho = 0$ and one of the revolutions (Fig. 2e). In both cases the polynomial $F(\rho)$ in (2.4) has a double root at zero and two other real roots $\rho_{1,2} = (1 + 2\mu \pm \sqrt{(1 + 4\mu)})/2$ ($\rho_2 < \rho_1$). The motion (oscillation or revolution) takes place with $\rho_2 \leqslant \rho \leqslant \rho_1$. Equation (2.3) may be integrated in terms of elementary functions, the result being [7]

$$\rho(\tau) = 2\mu^2 \frac{1 + 2\mu - \sqrt{1 + 4\mu}\cos 3\mu\tau}{(1 + 2\mu)^2 \sin^2 3\mu\tau + 4\mu^2 \cos^2 3\mu\tau}$$

where we have assumed that $\rho(0) = \rho_2$.

At the point $h = 0$, $\mu = 0$ we have (Fig. 2c) the unstable equilibrium $\rho = 0$, and motion along the separatrix defined by

$$\rho(\tau) = \frac{4}{4 + 9\tau^2}, \quad \theta(\tau) = \frac{1}{3}\left[(2m - 1)\pi + \operatorname{acrtg}\frac{3\tau}{2}\right], \quad m = 1, 2, 3$$

At the points of the curve $h = h(\mu_1)$ (Fig. 3), where $-9/32 < \mu < -1/4$, we have two types of motion (see Fig. 2c): stable equilibria, at which $\rho = \rho_*$, and oscillation in the neighbourhood of the equilibrium $\rho = 0$. In that case the polynomial $F(\rho)$ in (2.3) has a double root $\rho = \rho_*$ and two further real roots ρ_1 and ρ_2, where $\rho_2 < \rho_1 < \rho_*$. The oscillation in the neighbourhood of $\rho = 0$ occurs for $\rho_2 \leqslant \rho \leqslant \rho_1$. Equation (2.3) may be integrated in terms of elementary functions, the result being [7]

$$\rho(\tau) = \frac{\rho_1(\rho_* - \rho_2)\sin^2 u + \rho_2(\rho_* - \rho_1)\cos^2 u}{(\rho_* - \rho_2)\sin^2 u + (\rho_* - \rho_1)\cos^2 u}, \quad u = \frac{3}{2}[(\rho_* - \rho_1)(\rho_* - \rho_2)]^{1/2}\tau$$

where we have assumed that $\rho(0) = \rho_2$.

At the points of the curve $h = h_2(\mu)$ ($\mu > -9/32$, $\mu \neq 0$) we have unstable equilibria, for which $\rho = \rho_{**}$, and motion along the separatrices (Fig. 2c, e). The polynomial $F(\rho)$ then has a double root $\rho = \rho_{**}$ and two further real roots ρ_1 and ρ_2, where $\rho_2 < \rho_{**} < \rho_1$. If $-9/32 < \mu < 0$ (Fig. 2c), the value of ρ as the motion proceeds along the heteroclinic and homoclinic asymptotic trajectories varies in the ranges $\rho_2 \leqslant \rho < \rho_{**}$ and $\rho_{**} < \rho \leqslant \rho_1$, respectively. In both cases the function $\rho(\tau)$ is given implicitly by the following expression [7]

$$\ln^2 \frac{[\sqrt{(\rho_1 - \rho)(\rho_{**} - \rho_2)} + \sqrt{(\rho - \rho_2)(\rho_1 - \rho_{**})}]^2}{(\rho_1 - \rho_2)|\rho_{**} - \rho|} = 9(\rho_1 - \rho_{**})(\rho_{**} - \rho_2)\tau^2 \tag{2.11}$$

where $\rho(0) = \rho_2$ for the heteroclinic and $\rho(0) = \rho_1$ for the homoclinic asymptotic trajectories; $\rho \to \rho_{**}$ as $\tau \to \pm\infty$.

When $\mu > 0$ (Fig. 2e) the function $\rho(\tau)$ on the inner ($\rho_2 \leqslant \rho < \rho_{**}$) and outer ($\rho_{**} < \rho \leqslant \rho_1$) heteroclinic asymptotic trajectories is also given by (2.11).

Finally, the common point $(-9/32, 27/4096)$ of the curves $h = h_1(\mu)$ and $h = h_2(\mu)$ corresponds (Fig. 2b) to unstable equilibria of the system with $\rho = 9/64$ and motion along the separatrix defined by

$$\rho(\tau) = \left(\frac{81}{64}\tau^2 + 4\right)(256 + 9\tau^2)^{-1}, \quad \cos 3\theta = \left(\frac{27}{4096} - \frac{9}{32}\rho - \rho^2\right)\rho^{-3/2}$$

3. NON-LINEAR OSCILLATIONS OF THE PERTURBED SYSTEM

We shall now investigate how the results of our investigation of the unperturbed system with Hamiltonian γ_0 extend to the full system with Hamiltonian γ (see (1.8)).

3.1. By Poincaré's theory of periodic motions [8], from every equilibrium position of the unperturbed system (excluding the case $\mu = -9/32$) other than the origin, if e is sufficiently small, a unique solution of the full system arises which is T-periodic as a function of τ and analytic as a function of e. This corresponds in the original variables to 6π-periodic motion of the satellite.

The unstable equilibrium positions of system (2.1) corresponding to the equilibrium value $\rho = \rho_{**}$ transfer to unstable periodic solutions of the full system; this follows from the continuity with respect to e of the characteristics exponents of the corresponding linear equations of perturbed motion.

Let us investigate the stability of periodic motions originating in the manner just described from stable equilibrium positions (corresponding to $\rho = \rho_*$) of the unperturbed system. To that end, we first normalize the Hamiltonian γ_0 in the neighbourhood of these equilibria. Let $\theta = \theta_* + x$ (where θ_* is the equilibrium value of the angle θ), $\rho = \rho_* + y$. Then the Hamiltonian γ_0 may be expanded in series

$$\gamma_0 = \gamma_0^{(2)} + \gamma_0^{(3)} + \gamma_0^{(4)} + \ldots \tag{3.1}$$

$$\gamma_0^{(2)} = \frac{9}{2}\rho_*^{3/2}x^2 + \left(1 - \frac{3}{8}\rho_*^{-1/2}\right)y^2, \quad \gamma_0^{(3)} = \frac{27}{4}\rho_*^{1/2}x^2 y + \frac{1}{16}y^3\rho_*^{-3/2}$$

$$\gamma_0^{(4)} = -\frac{27}{8}\rho_*^{3/2}x^4 + \frac{27}{16}\rho_*^{-1/2}x^2y^2 - \frac{3}{128}\rho_*^{-5/2}y^4$$

The dots stand for terms of order more than four in x and y.

The change of variables $x = x_1/\alpha$, $y = \alpha y_1$, $\alpha = \sqrt{(6)}\rho_*^{1/2}/(8\rho_*^{1/2} - 3)^{1/4}$ normalizes the quadratic part of (3.1), bringing it to the form $1/2\Omega(x_1^2 + y_1^2)$, $\Omega = 3/2\rho_*^{1/2}(8\rho_*^{1/2} - 3)^{1/2}$. Next, applying a canonical transformation $x_1, y_1 \to \xi, \eta$ of the Birkhoff type, we eliminate third-order terms in the Hamiltonian and simplify the fourth-order terms. The required normal form of the Hamiltonian γ_0 in the neighbourhood of the equilibrium under consideration is

$$G = \frac{1}{2}\Omega(\xi^2 + \eta^2) + \frac{1}{4}c(\xi^2 + \eta^2)^2 + \ldots \tag{3.2}$$

$$c = -\frac{9}{64}[z(z-3)^2 + 3(9z^2 + 6z + 5)]\rho_*^{-1/2}z^{-2}, \quad z = 8\rho_*^{1/2} - 3$$

The quantity z in (3.2) is positive in the domain $\mu > -9/32$ of existence of stable equilibrium positions; hence $c < 0$.

If we now normalize the Hamiltonian γ in the neighbourhood of the periodic solution of the complete system generated by the stable equilibrium under consideration, we obtain a Hamiltonian of the form (3.2) with the coefficients Ω and c corrected by quantities of the order of e. For sufficiently small e, it follows from the inequality $c < 0$ that the condition for the Hamiltonian γ to be non-degenerate is satisfied in the neighbourhood of the periodic solution. Hence, by the Arnol'd–Moser theorem [9, 10], the solution in question is stable in Lyapunov's sense.

3.2. We will now show that motions of the complete system, beginning in some finite neighbourhood of the origin, are bounded, and estimate the size of that neighbourhood.

We introduce action-angle variables I, w in the domain of revolutions of the unperturbed system [11], setting

$$I(h) = (2\pi)^{-1}\oint\rho(\theta, h)d\theta \tag{3.3}$$

where the integration is performed along the closed trajectory $\rho = \rho(\theta, h)$ defined by (2.3). The inverse function to (3.3), $h = h(I)$, is the Hamiltonian γ_0 written in action-angle variables.

Let us estimate the size of the neighbourhood of the origin outside which the non-degeneracy condition $d^2h/dI^2 \neq 0$ holds in the domain of revolutions. Using (3.3) and (2.3), it can be shown that

$$\frac{d^2h}{dI^2} = \frac{\omega^3}{2\pi}\oint\frac{\partial^2\gamma_0/\partial\rho^2}{(\partial\gamma_0/\partial\rho)^3}d\theta = \frac{\omega^3}{2\pi}\int_0^{2\pi}\frac{8\rho^{1/2} + 3\cos 3\theta}{\rho^{1/2}(2\rho + \frac{3}{2}\rho^{1/2}\cos 3\theta - \mu)^3}d\theta \tag{3.4}$$

Since the denominator of the fraction in (3.4) is positive in the domain of revolutions ($\partial\gamma_0/\partial\rho = d\theta/d\tau > 0$), it follows that the integrand is positive on revolution-trajectories (RTs) on which $\rho > 9/64$. When that happens, $d^2h/dl^2 > 0$, and the non-degeneracy condition holds.

It can be shown that when $\mu < 27/32$ a RT passing through the point $\rho = 9/64$, $\theta = 0$ exists. On this RT, $9/64 \leqslant \rho \leqslant R_1$, where R_1, where R_1 is the unique real root (except in the case $\mu = 33/64$) of the equation

$$\rho^2 - \rho^{3/2} - \mu\rho = \frac{9}{64}\left(\frac{33}{64} - \mu\right) \tag{3.5}$$

in the domain under consideration. If $\mu = 33/64$, corresponding to $h = 0$, this equation has three roots: 0, 9/64 and 121/64; then $R_1 = 121/64$.

Since the non-degeneracy condition holds on this RT, it follows from Moser's invariant curve theorem [10] that the mapping generated by motions of the perturbed system over the period T has an invariant curve near the RT under consideration, provided that e is sufficiently small. For all trajectories of the perturbed system starting inside this curve, one has $\rho(\tau) < R_1(1 + O(e))$.

Now let $\mu \geqslant 27/32$. Since then $\rho_{**} \geqslant 9/64$, it follows that the circle $\rho = 9/64$ does not intersect the domain of revolutions, and the non-degeneracy condition holds for all RTs. Hence, by Moser's theorem, invariant curves exist. Choose one of them, say, close to a RT, on which ρ does not take values exceeding $2R_2$, where R_2 is the maximum value of ρ on the separatrix (see Fig. 2e). The quantity R_2 is the unique root in the domain under consideration of the equation

$$\rho^2 - \rho^{3/2} - \mu\rho = \rho_{**}^2 + \rho_{**}^{3/2} - \mu\rho_{**} \tag{3.6}$$

For all trajectories beginning inside the invariant curve, we have $\rho(\tau) < 2R_2(1 + O(e))$.

4. NON-LINEAR OSCILLATIONS OF A SATELLITE

The conclusions reached in Section 3 concern the fairly wide class of systems described by a Hamiltonian of the form (1.8). We will now apply the results to the problem of plane non-linear oscillations of a satellite.

If one has equilibrium positions of the unperturbed system with Hamiltonian γ_0 (we again assume that $\mu \neq -9/32$, which in the original notation corresponds to $\beta \neq -27/1024$) corresponding to the same equilibrium value of ρ but distinct values of θ (other than $2\pi/3$), then the 6π-periodic solutions of Eq. (1.1) generated by these equilibria develops into one another when ν is varied by 2π and 4π. These motions correspond to the same 6π-periodic motion of the satellite, which occurs near the eccentricity oscillations (1.2).

Corresponding to the boundary value $\beta = -27/1024$ between λ domains with a different number of periodic motions, one has the following bifurcation curve in the e, ω_0 plane

$$\omega_0 = 1/3 + (2137/5120)e^2 + O(e^4) \tag{4.1}$$

which is shown by the dashed curve in Fig. 1. For parameter values e and ω_0 such that the point (e, ω_0) is under the curve (4.1), there are no 6π-periodic motions of the satellite distinct from oscillations (1.2).

For points (e, ω_0) between the curve (4.1) and the resonance curve (1.5), two 6π-periodic motions of the satellite distinct from (1.2) exist. They are generated from the equilibrium positions of the unperturbed system corresponding to $\rho = \rho_*$ and $\rho = \rho_{**}$, and are described by relations of the form

$$\psi(\nu) = -\frac{9}{4}e\sin\nu - \frac{3}{2}e\sqrt{\sigma}\sin\frac{\nu}{3} + O(e^2) \tag{4.2}$$

where $\sigma = \rho_*$ and $\sigma = \rho_{**}$, respectively. According to Section 3, the first of these motions is stable and the other unstable.

For values of e and ω_0 on the resonance curve (1.5) a unique (distinct from (1.2)) 6π-periodic motion of the satellite exists, described by an equation of the form (4.2). This motion is stable. The periodic motion corresponding to $\rho = \rho_{**}$ is identical with the eccentricity oscillations (1.2) and is unstable.

If the point (e, ω_0) lies above the resonance curve (1.5), the motion corresponding to $\rho = \rho_{**}$ is again separated from the eccentricity oscillations (1.2) and we have two (distinct from (1.2)) 6π-periodic motions of the satellite, described by equations of type (4.2); one is stable—for $\sqrt{\sigma} = \sqrt{\rho_*}$—and the other unstable—for $\sqrt{\sigma} = -\sqrt{\rho_{**}}$.

Any motions $\psi = \psi(\nu)$ of the satellite beginning sufficiently close to its eccentricity oscillations $\psi = \psi^*(\nu)$ remain in a finite neighbourhood of those oscillations. An estimate of the size of that neighbourhood was given in Section 3: $|\psi(\nu) - \psi^*(\nu)| \leqslant 3/2e\xi(1 + O(e))$, where $\xi = \sqrt{R_1}$ if $\beta < 81/1024$ and $\xi = \sqrt{(2R_2)}$, if $\beta \geqslant 81/1024$.

Here the quantities $R_i = R_i(\mu) = R_i((32/3)\beta)$ $(i = 1, 2)$ are the roots of Eqs (3.5) and (3.6), respectively, while β for the relevant e and ω_0 values is found from the equation

$$\omega_0 = {}^1\!/_3 + e^2(71/160 + \beta) + O(e^4)$$

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